Journal of Sound and Vibration (2003) **259**(4), 935–953
doi:10.1006/jsvi.2002.5138, available online at http://www.idealibrary.com on IDEAL®

SENSOR ARRAY DESIGN FOR WAVE DECOMPOSITION IN THE PRESENCE OF COUPLED MOTION

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(*Received* 29 *June* 2000, *and in final form* 8 *April* 2002)

The estimation of wave amplitudes is investigated for the case in which two different vibration types couple to the measured variables. Particular reference is made, by way of example, to the coupling of axial wall motion to pressure waves in a fluid-filled pipe. Two different types of sensor array are considered: one in which the same parameter is measured at a number of different locations, and one in which the measured parameters are not all identical. Predictions of variation in frequency range and sensitivity to measurement errors are made based on the behaviour of the determinant of the transformation matrix, and are verified by simulation. It is shown that the analytical expression for this determinant can give valuable insight into the design of sensor arrays, and that the choice of measured variables is critical to achieving low sensitivity to measurement error.

1. INTRODUCTION

Reducing the transmission of structure-borne noise and vibration can be a challenging task. Typically, a variety of different vibration types can be present in a structural element, and the relative importance of these is not generally known a priori. However, this information is important to the analyst, as the effectiveness of a particular vibration treatment depends strongly on the vibration types that are present. Wave decomposition, in which the amplitudes of the various waves are estimated from vibration measurements, can therefore provide valuable insight.

The accuracy of the estimates of wave amplitude depends strongly on the specific measurement environment. In general, however, there is often considerable uncertainty in the assumed material properties, sensor locations, etc. and there are inevitably errors in the measured data. A low sensitivity to measurement and other errors is therefore a prerequisite of a reliable measurement system.

In simple structural elements, the measured variables can be chosen so that, to a good approximation, they are influenced only by a single vibration type, and the estimation of wave amplitudes can therefore be performed for each vibration type independently. The design of appropriate sensor arrays for these situations has been investigated in previous work [1] and is summarized in section 4.

In more complicated structural elements, the measured variables can be strongly influenced by more than one vibration type. Examples of this include coupled flexural and extensional motion in curved beams, and the coupling of axial and circumferential motion in a pipe wall with pressure variations in the fluid. Under these circumstances, the amplitudes of the waves associated with the different vibration types should ideally be estimated simultaneously. The design of appropriate sensor arrays for these applications is more complicated, and a strategy to assist in the design process is the subject of this paper.

In the following section, the principle of estimating wave amplitudes from a set of simultaneous measurements is described. This is followed by a brief discussion regarding the sensitivity of the estimates to errors in the assumed parameters and experimental measurements. In section 4, the problem of designing sensor arrays for wave decomposition is discussed, and the use of the transformation matrix determinant as a design guide is proposed. Sections 5 and 6 illustrate how the determinant can be used in the design of single parameter and hybrid arrays respectively. Simulations of measurements in fluid-filled pipes are used as examples, and sensitivity to errors is demonstrated by the introduction of random noise to the simulations.

2. PRINCIPLES OF WAVE DECOMPOSITION

Consider a response parameter $\psi(z)$, such as acceleration or strain, of a vibrating structure. Typically, there will be a number of waves present that will influence this variable. It can, therefore, be expressed as the sum of the contributions of the individual waves in the form

$$\psi(z) = \Phi_1^+ q_1^+ \mathrm{e}^{-\mathrm{i}k_1 z} + \Phi_1^- q_1^- \mathrm{e}^{\mathrm{i}k_1 z} + \ldots + \Phi_m^+ q_m^+ \mathrm{e}^{-\mathrm{i}k_m z} + \Phi_m^- q_m^- \mathrm{e}^{\mathrm{i}k_m z}, \tag{1}$$

where Φ_r^+ and Φ_r^- are the positive- and negative-going wave amplitudes and k_r is the wavenumber, the subscript r denoting the rth wave type. The terms q_r^+ and q_r^- are conversion factors that relate the wave amplitudes to the measured variable. It should be noted that the wave amplitudes can be described in terms of *any* parameter that varies due to the presence of the wave. For example, axial waves in a rod can be described in terms of axial displacement, axial strain, axial stress, or even the circumferential strain that results from the Poisson effect. If a particular wave type, r, is described in terms of the response parameter, $\psi(z)$, then $q_r^+ = q_r^- = 1$. If, however, the wave type is described in terms of some other parameter, the relevant theoretical conversion factors, q_r^+ and q_r^-, must be applied.

If n simultaneous measurements $\psi_s(z_s)$ are taken on the structure they can be written in matrix form as

$$\mathbf{\Psi} = \mathbf{F}\mathbf{\Phi} \tag{2}$$

where

$$\mathbf{\Psi}^{\mathbf{T}} = (\psi_1(z_1) \;\; \psi_2(z_2) \ldots \psi_n(z_n)), \qquad \mathbf{\Phi}^{\mathbf{T}} = (\Phi_1^+ \;\; \Phi_1^- \;\; \ldots \Phi_m^+ \;\; \Phi_m^-),$$

$$\mathbf{F} = \begin{bmatrix} q_{11}^+ \mathrm{e}^{-\mathrm{i}k_1 z_1} & q_{11}^- \mathrm{e}^{\mathrm{i}k_1 z_1} & \cdots & q_{1m}^+ \mathrm{e}^{-\mathrm{i}k_m z_1} & q_{1m}^- \mathrm{e}^{\mathrm{i}k_m z_1} \\ q_{21}^+ \mathrm{e}^{-\mathrm{i}k_1 z_2} & q_{21}^- \mathrm{e}^{\mathrm{i}k_1 z_2} & \cdots & q_{2m}^+ \mathrm{e}^{-\mathrm{i}k_m z_2} & q_{2m}^- \mathrm{e}^{\mathrm{i}k_m z_2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ q_{n1}^+ \mathrm{e}^{-\mathrm{i}k_1 z_n} & q_{n1}^- \mathrm{e}^{\mathrm{i}k_1 z_n} & \cdots & q_{nm}^+ \mathrm{e}^{-\mathrm{i}k_m z_n} & q_{nm}^- \mathrm{e}^{\mathrm{i}k_m z_n} \end{bmatrix}. \tag{3a, b, c}$$

In order to estimate the wave amplitudes from these measurements it is necessary, as a minimum requirement, for the number of measurements, n, to equal the number of waves assumed present ($2\,m$). The transformation matrix $\mathbf{F}$ is then square, and the wave amplitudes can be estimated using matrix inversion to give

$$\mathbf{\Phi} = \mathbf{F}^{-1}\boldsymbol{\psi} \tag{4}$$

provided that matrix $\mathbf{F}$ is non-singular. If the number of simultaneous measurements exceeds the number of waves assumed to influence the measurements, the problem is overdetermined. A "best" estimate of the wave amplitudes can then be found, in a least-squares sense, using the Moore–Penrose inverse, giving

$$\mathbf{\Phi} = (\mathbf{F}^{\mathrm{H}}\mathbf{F})^{-1}\mathbf{F}^{\mathrm{H}}\boldsymbol{\psi}, \tag{5}$$

where $(\ )^{\mathrm{H}}$ denotes the conjugate transpose of a matrix [2]. However, while the use of an overdetermined system is certainly desirable in terms of reducing errors, it significantly increases the complexity and cost of the measurement system. For this reason, in the work described in this paper, **F** is assumed to be a square matrix.

3. ERRORS AND CONDITIONING

In practice, there will inevitably be errors in both the vector of experimental measurements, $\boldsymbol{\psi}$, and the transformation matrix, **F**. These errors will result both from experimental error (in the measured values, ψ_s, and in the sensor placement, z_s) and from imperfect knowledge of the structural properties and behaviour (causing errors in the assumed wavenumbers, k, and conversion factors, $q^{\pm}$). The sensitivity of the wave amplitude estimate to these errors is determined by the *conditioning* of the problem, with a well-conditioned problem being relatively insensitive to errors in the input data. This conditioning can be quantified by the two-norm condition number of the transformation matrix **F**, with a condition number of unity indicating optimal conditioning and a condition number of infinity signifying that the matrix is singular.

It can be shown that if there exists an error, $\boldsymbol{\delta}\mathbf{F}$, in the transformation matrix such that

$$\mathbf{F}' = \mathbf{F} + \boldsymbol{\delta}\mathbf{F}, \tag{6}$$

then the resulting error, $\boldsymbol{\delta\Phi}$, in the wave amplitude estimate is related to $\delta\mathbf{F}$ by

$$\frac{||\boldsymbol{\delta\Phi}||}{||\mathbf{\Phi}||} \leqslant \kappa(\mathbf{F})\frac{||\boldsymbol{\delta}\mathbf{F}||}{||\mathbf{F}||}, \tag{7}$$

where $||\ ||$ denotes the norm of a vector or matrix and $\kappa(\mathbf{F})$ is the condition number of the matrix **F** [3]. Similarly, if there exists an error, $\boldsymbol{\delta\psi}$, in the measured variables such that

$$\boldsymbol{\psi}' = \boldsymbol{\psi} + \boldsymbol{\delta\psi}, \tag{8}$$

the resulting error, $\boldsymbol{\delta\Phi}$, in the wave amplitude estimate is related to $\boldsymbol{\delta\psi}$ by

$$\frac{||\boldsymbol{\delta\Phi}||}{||\mathbf{\Phi}||} \leqslant \kappa(\mathbf{F})\frac{||\boldsymbol{\delta\psi}||}{||\boldsymbol{\psi}||}. \tag{9}$$

The condition number of the transformation matrix, **F**, therefore determines the upper bound to the resulting errors in the wave amplitude estimate. Minimizing the condition number minimizes the maximum possible error, and thus allows the best estimate of the wave amplitudes from a given quality of input data.

The elements of the matrix, **F**, and therefore its condition number, are determined by the measured variables, the measurement locations, and the parameters used to describe the waves. The sensitivity of the wave amplitude estimate to errors can therefore be minimized through the use of an appropriate sensor array for the measurements. In practice, however, it is generally only possible to achieve good conditioning over a limited frequency band. It is therefore necessary to consider the variation of conditioning with frequency when designing a sensor array for a given measurement application.

4. SENSOR ARRAY DESIGN

Overall, the goal in designing the sensor array is to obtain an accurate and reliable estimate of the wave amplitudes that are present. Clearly, the quality of the estimate depends not only on the conditioning of the problem, as discussed in the previous section, but also on the quality of the measured data. This introduces additional considerations, such as the relative accuracy of different sensor types, and their susceptibility to interference from other sources. However, in the interests of generality, it is assumed in what follows that all sensors have similar accuracy, and sensor arrays are compared only in terms of the conditioning achieved.

The design of sensor arrays for wave amplitude estimation in situations in which only a single vibration type affects the measured variables has been covered in previous work. In summary, for conventional sensor arrays, in which a single variable is measured at uniformly spaced locations, and the number of sensors is equal to the number of waves assumed present (i.e., a fully determined, rather than overdetermined, system) optimal conditioning occurs at a sensor spacing of approximately a quarter wavelength, with singularity occurring at a half-wavelength spacing. It has also been shown that hybrid sensor arrays, in which more than one variable is measured, can result in improved conditioning in cases where the array size is restricted. A typical example of this is the measurement of flexural wave amplitude between two closely spaced discontinuities. Under these circumstances, an array comprising of a strain gauge and an accelerometer at each of two locations can offer substantially better conditioning than an array of four uniformly spaced accelerometers [1].

When the measured variables are influenced by more than one vibration type, the design of an appropriate sensor array becomes more complicated. It is still desirable to minimize sensitivity to errors, and thus a low condition number is a suitable indicator of potential performance. However, the selection of appropriate sensors and sensor locations to achieve this goal is not intuitive. In what follows, the determinant of matrix **F** is expanded, and its behaviour is used to determine the qualitative changes in conditioning that result from specific changes in sensor spacing. It should be noted that good conditioning, as indicated by a low condition number in the frequency range of interest, is still used as a measure of sensor array suitability. The determinant is simply used as an aid in selecting sensor types and spacings to obtain a well-conditioned calculation.

Consider the case where there are two vibration types, each comprising of a pair of propagating waves that affect the measured variable. The output of any sensor, being the sum of the contributions of the individual waves can be written as

$$\psi(z) = q_1^+ \Phi_1^+ \mathrm{e}^{-\mathrm{i}k_1 z} + q_1^- \Phi_1^- \mathrm{e}^{\mathrm{i}k_1 z} + q_2^+ \Phi_2^+ \mathrm{e}^{-\mathrm{i}k_2 z} + q_2^- \Phi_2^- \mathrm{e}^{\mathrm{i}k_2 z}. \tag{10}$$

If the sensor array comprises four sensors, the relationship between the sensor outputs and wave amplitudes can be expressed as

$$\boldsymbol{\psi} = \mathbf{F}\boldsymbol{\Phi}, \tag{11}$$

where

$$\boldsymbol{\psi}^{\mathrm{T}} = (\psi_1(z_1)\ \ \psi_2(z_2)\ \ \psi_3(z_3)\ \ \psi_4(z_4)), \qquad \boldsymbol{\Phi}^{\mathrm{T}} = (\Phi_1^+\ \ \Phi_1^-\ \ \Phi_2^+\ \ \Phi_2^+),$$

$$\mathbf{F} = \begin{bmatrix} q_{11}^+ \mathrm{e}^{-\mathrm{i}k_1 z_1} & q_{11}^- \mathrm{e}^{\mathrm{i}k_1 z_1} & q_{12}^+ \mathrm{e}^{-\mathrm{i}k_2 z_1} & q_{12}^- \mathrm{e}^{\mathrm{i}k_2 z_1} \\ q_{21}^+ \mathrm{e}^{-\mathrm{i}k_1 z_2} & q_{21}^- \mathrm{e}^{\mathrm{i}k_1 z_2} & q_{22}^+ \mathrm{e}^{-\mathrm{i}k_2 z_2} & q_{22}^- \mathrm{e}^{\mathrm{i}k_2 z_2} \\ q_{31}^+ \mathrm{e}^{-\mathrm{i}k_1 z_3} & q_{31}^- \mathrm{e}^{\mathrm{i}k_1 z_3} & q_{32}^+ \mathrm{e}^{-\mathrm{i}k_2 z_3} & q_{32}^- \mathrm{e}^{\mathrm{i}k_2 z_3} \\ q_{41}^+ \mathrm{e}^{-\mathrm{i}k_1 z_4} & q_{41}^- \mathrm{e}^{\mathrm{i}k_1 z_4} & q_{42}^+ \mathrm{e}^{-\mathrm{i}k_2 z_4} & q_{42}^- \mathrm{e}^{\mathrm{i}k_2 z_4} \end{bmatrix}. \tag{12a, b, c}$$

If the estimation of wave amplitudes is to be relatively insensitive to errors, it is necessary that $\mathbf{F}$ be a well-conditioned matrix. This can be achieved by the appropriate selection of measured variables and wave descriptors, which between them determine $q_{ij}^{\pm}$, and measurement locations, z_i. However, the effect on the conditioning of varying these parameters is not readily apparent, and thus, without additional insight, refinement of sensor array design is largely a matter of trial and error.

Some additional insight can be gained by considering the behaviour of the determinant of matrix $\mathbf{F}$. However, the general expression for the determinant of a 4×4 matrix is relatively complicated, and therefore some degree of simplification is desirable at this stage. Initially, this involves the assumption that the sensors are used in pairs of a particular type. For example, a sensor array might comprise of two pairs of accelerometers, or of a pair of accelerometers and a pair of strain gauges. It is assumed, with no loss of generality, that sensors 1 and 2 constitute the first sensor pair, while sensors 3 and 4 constitute the second sensor pair. Under these circumstances, matrix $\mathbf{F}$ becomes

$$\mathbf{F} = \begin{bmatrix} q_{11}^{+}\mathrm{e}^{-\mathrm{i}k_1 z_1} & q_{11}^{-}\mathrm{e}^{\mathrm{i}k_1 z_1} & q_{12}^{+}\mathrm{e}^{-\mathrm{i}k_2 z_1} & q_{12}^{-}\mathrm{e}^{\mathrm{i}k_2 z_1} \\ q_{11}^{+}\mathrm{e}^{-\mathrm{i}k_1 z_2} & q_{11}^{-}\mathrm{e}^{\mathrm{i}k_1 z_2} & q_{12}^{+}\mathrm{e}^{-\mathrm{i}k_2 z_2} & q_{12}^{-}\mathrm{e}^{\mathrm{i}k_2 z_2} \\ q_{21}^{+}\mathrm{e}^{-\mathrm{i}k_1 z_3} & q_{21}^{-}\mathrm{e}^{\mathrm{i}k_1 z_3} & q_{22}^{+}\mathrm{e}^{-\mathrm{i}k_2 z_3} & q_{22}^{-}\mathrm{e}^{\mathrm{i}k_2 z_3} \\ q_{21}^{+}\mathrm{e}^{-\mathrm{i}k_1 z_4} & q_{21}^{-}\mathrm{e}^{\mathrm{i}k_1 z_4} & q_{22}^{+}\mathrm{e}^{-\mathrm{i}k_2 z_4} & q_{22}^{-}\mathrm{e}^{\mathrm{i}k_2 z_4} \end{bmatrix}. \tag{13}$$

The expression for the determinant of $\mathbf{F}$ depends on the relative signs of the conversion factors, q, with $q_{ij}^{-} = \pm q_{ij}^{+}$. If $q_{ij}^{+} = q_{ij}^{-}$ for all ij then the determinant is given by

$$\begin{aligned} \det(\mathbf{F}) = 4[&-q_{11}^{+}q_{11}^{+}q_{22}^{+}q_{22}^{+}\sin(k_1(z_2 - z_1))\sin(k_2(z_4 - z_3)) \\ &- q_{12}^{+}q_{12}^{+}q_{21}^{+}q_{21}^{+}\sin(k_1(z_4 - z_3))\sin(k_2(z_2 - z_1)) \\ &+ q_{11}^{+}q_{12}^{+}q_{21}^{+}q_{22}^{+}(\sin(k_1(z_3 - z_1))\sin(k_2(z_4 - z_2)) - \sin(k_1(z_4 - z_1))\sin(k_2(z_3 - z_2)) \\ &- \sin(k_1(z_3 - z_2))\sin(k_2(z_4 - z_1)) + \sin(k_1(z_4 - z_2))\sin(k_2(z_3 - z_1)))], \end{aligned} \tag{14}$$

while if $q_{11}^{-} = -q_{11}^{+}$, $q_{12}^{-} = -q_{12}^{+}$, $q_{21}^{-} = q_{21}^{+}$, and $q_{22}^{-} = q_{22}^{+}$, the determinant becomes

$$\begin{aligned} \det(\mathbf{F}) = 4[&q_{11}^{+}q_{11}^{+}q_{22}^{+}q_{22}^{+}\sin(k_1(z_2 - z_1))\sin(k_2(z_4 - z_3)) \\ &+ q_{12}^{+}q_{12}^{+}q_{21}^{+}q_{21}^{+}\sin(k_1(z_4 - z_3))\sin(k_2(z_2 - z_1)) \\ &+ q_{11}^{+}q_{12}^{+}q_{21}^{+}q_{22}^{+}(-\cos(k_1(z_3 - z_1))\cos(k_2(z_4 - z_2)) \\ &+ \cos(k_1(z_4 - z_1))\cos(k_2(z_3 - z_2)) \\ &+ \cos(k_1(z_3 - z_2))\cos(k_2(z_4 - z_1)) - \cos(k_1(z_4 - z_2))\cos(k_2(z_3 - z_1)))]. \end{aligned} \tag{15}$$

It is apparent from these expressions that the determinant can be expressed as the superposition of a number of modulated harmonic terms. The frequencies of variation and modulation with wavenumber are determined by the separations of the individual sensors in terms of the wavelengths of the two wave types present. In order to determine the upper bound to the upper frequency limit of the sensor array, it is necessary to identify the first (non-zero frequency) zero of the determinant. Furthermore, the behaviour of the determinant below that frequency can give an indication of the variation of conditioning with frequency, with a large determinant generally being associated with a well-conditioned system.

The precise behaviour of the determinant depends strongly on the form of the sensor array and on the relative values of the two wavenumbers, k_1 and k_2. It is thus difficult to establish general rules for the design of sensor arrays in this application. However, valuable insight can be gained by considering specific situations. In the following sections, two particular sensor array types will be considered: one in which the four sensors measure

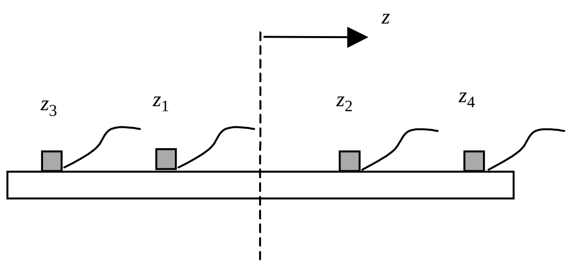

Figure 1. Sensor array configuration.

the same variable (a single parameter array), and one in which two measurements of one variable and two measurements of a second variable are taken (a hybrid array). In both cases, the sensor array is assumed symmetrical with the sensor locations as shown in Figure 1. Predictions are made, based on the behaviour of the determinant, regarding the conditioning of the wave amplitude estimation, and simulations are performed to demonstrate the effects on the useable frequency range and sensitivity to noise.

5. WAVE AMPLITUDE ESTIMATION USING A SINGLE PARAMETER ARRAY

In this type of array, a single variable, e.g., axial acceleration, is measured at four different locations simultaneously. For simplicity, that same variable is used to describe the wave amplitudes. (Note that these waves can be expressed in terms of any other variable simply by applying the appropriate conversion factor. There will, however, be an additional error associated with this conversion.) Under these circumstances $q_{ij}^{\pm} = 1$ for all ij, and the determinant can be written as

$$\det(\mathbf{F}) = (Q + R + S + T), \tag{16}$$

where

$$Q = -4\sin(k_1(z_2 - z_1))\sin(k_2(z_4 - z_3)),$$

$$R = -4\sin(k_1(z_4 - z_3))\sin(k_2(z_2 - z_1)),$$

$$S = -8\sin(k_1(z_3 - z_1))\sin(k_2(z_3 - z_1)),$$

$$T = 8\sin(k_1(z_4 - z_1))\sin(k_2(z_4 - z_1)), \tag{17}$$

An intuitive approach to designing the sensor array would suggest that, at any given frequency, one pair of sensors should be spaced to efficiently sense the presence of each wave type. This would mean that, for the sensor array shown in Figure 1, sensors 1 and 2 would be placed to suit the larger wavenumber (assumed to be k_1), while sensors 3 and 4 would be placed to suit the smaller wavenumber (assumed to be k_2). If this is the case

$$k_1(z_2 - z_1) \approx k_2(z_4 - z_3). \tag{18}$$

This approximate relationship will be used as a starting point in all subsequent analyses.

The behaviour of the determinant of the transformation matrix, **F**, is strongly dependent on the relative values of the two wavenumbers k_1 and k_2. Three separate cases are therefore considered in the following subsections: when there is a relatively large difference between the wavenumbers (i.e., $k_1 \gg k_2$), when there is a relatively small difference between the wavenumbers (i.e. $k_1 \approx k_2$), and the special case where $k_2 = k_1/3$.

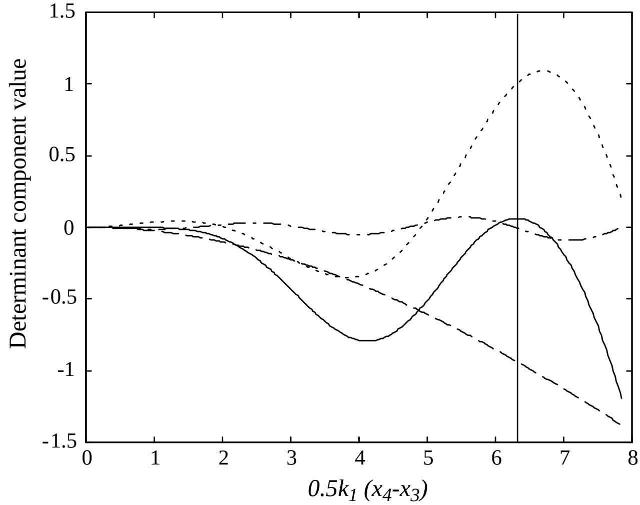

Figure 2. Terms contributing to determinant as a function of sensor spacing ($k_1 \gg k_2$): (——) Det(**F**); (– – – –) Q; (–·–·–) R; (· · · · · ·) $S+T$.

5.1. CASE 1: $k_1 \gg k_2$

Using the approximate sensor placement given in equation (18), if $k_1 \gg k_2$ then $k_2(z_2 - z_1)$ is small and thus the modulation of R with changing wavenumber is relatively slow (see Figure 2). This means that at low frequencies the magnitude of R is relatively small. At these frequencies the value of the determinant is therefore dominated by the remaining three terms, Q, S and T. The term Q varies comparatively slowly with wavenumber relative to S and T, and the variation of the determinant is dominated by the variation of the sum of S and T, which is $\sim \cos(0{\cdot}5k_1(z_4 - z_3))$. The first zero in the determinant approximately coincides with the first peak in the sum of S and T (i.e., when $0{\cdot}5k_1(z_4 - z_3) \approx 2\pi$). Singularity of the transformation matrix will therefore occur at a much lower frequency than if each wave type were to be sensed separately (i.e., when $k_1(z_2 - z_1) \ll \pi$). It should also be noted that within this frequency range, the separation of the inner pair of sensors $(z_2 - z_1)$, is very small in terms of wavelengths for either wave type. Increasing this separation slightly might therefore improve conditioning at low frequencies, without significantly affecting the upper frequency limit of the sensor array. Furthermore, adjustment of the frequency of singularity is most readily achieved by changing the separation of the outer pair of sensors $(z_4 - z_3)$. It is apparent that reducing this spacing will raise the frequency at which matrix singularity occurs and thus, in general, increase the upper frequency limit of the array. However, this will be accompanied by a deterioration of conditioning at lower frequencies.

An example of this situation is a steel pipe, having a 50 mm internal diameter and 5 mm wall thickness, containing compressed air. (Derivations of the required equations, along with the definitions of the symbols used, are given in the Appendices A and B.) Under these circumstances $k_f \approx 15k_e$, thus satisfying the requirement that $k_f \gg k_e$. Letting the separation of the inner sensor pair $(z_2 - z_1) = 0{\cdot}150$ m and the separation of the outer sensor pair $(z_4 - z_3) = 2{\cdot}25$ m satisfies the initial assumption that $k_f(z_2 - z_1) \approx k_e(z_4 - z_3)$. In Figure 3 is shown the variation of the condition number with frequency for this sensor array, together with a plot of $1 + 1/|\det(\mathbf{F})|$. It is apparent that $1 + 1/|\det(\mathbf{F})|$ qualitatively approximates the behaviour of the condition number. The qualitative

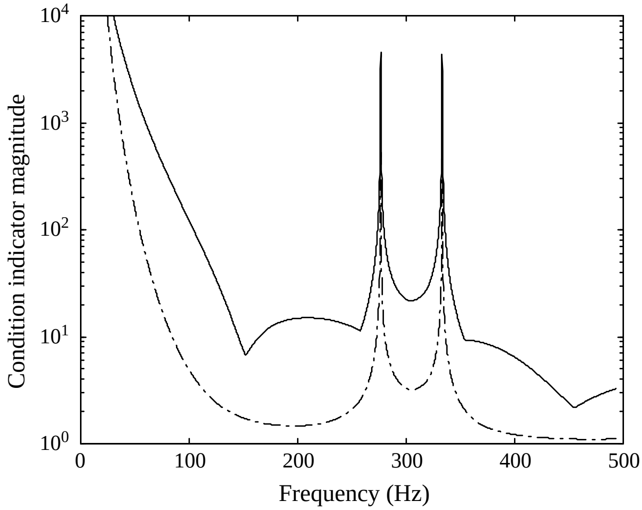

Figure 3. Estimates of matrix condition (steel/air): (——) two-norm condition number; (–·–·–) $1 + 1/|\det(\mathbf{F})|$.

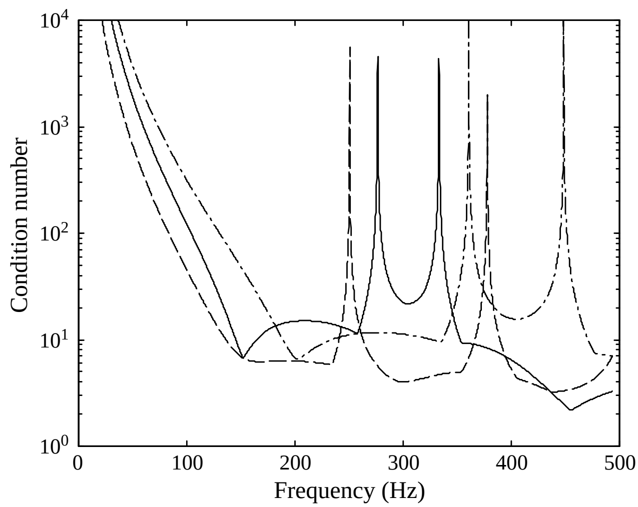

Figure 4. Matrix condition number (two-norm) (steel/air $k_f \approx 15k_e$): (——) $(z_2 - z_1) = 0{\cdot}150$ m, $(z_4 - z_3) = 2{\cdot}250$ m; (- - - - -) $(z_2 - z_1) = 0{\cdot}450$ m, $(z_4 - z_3) = 2{\cdot}250$ m; (–·–·–) $(z_2 - z_1) = 0{\cdot}150$ m, $(z_4 - z_3) = 1{\cdot}700$ m.

behaviour of the condition number with frequency can thus be predicted with acceptable accuracy from the behaviour of the determinant.

As predicted from examination of the expression for the determinant of matrix, **F**, the proposed sensor array has a relatively narrow frequency range, with the first singularity occurring at a frequency of approximately 275 Hz, corresponding to $0{\cdot}5k_f(z_4 - z_3) \approx 5{\cdot}7$. The effect of increasing the separation of the inner sensor pair, and of decreasing the outer sensor pair, on the conditioning can be seen in Figure 4. In the former case, the separation of the inner pair of sensors is tripled, so that $(z_2 - z_1) = 0{\cdot}450$ m and $k_f(z_2 - z_1) \approx 3k_e(z_4 - z_3)$. It can be seen that the effect of this modification to the sensor array is to improve the conditioning at low frequencies, at the expense of a slightly reduced upper frequency limit, as predicted. In the latter case, the separation of the outer pair of sensors is reduced to give $(z_4 - z_3) = 1{\cdot}7$ mm, so that $k_f(z_2 - z_1) \approx 1{\cdot}3k_e(z_4 - z_3)$. The first

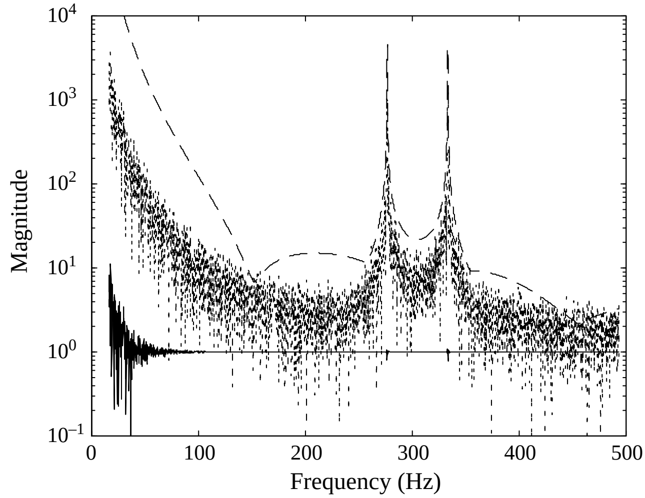

Figure 5. Condition number and normalized wave amplitude estimate (steel/air): (——) extensional wave; (......) fluid wave; (--------) condition number.

singularity is then seen to occur at approximately 360 Hz, at the expense of poorer conditioning at low frequencies.

The sensitivity of the wave decomposition to measurement errors can be demonstrated by simulation. In this simulation it is assumed that there exist only positive-going waves of each type, with their amplitudes such that they contribute equally to the net energy flow. The resulting measured parameters are calculated, and the "measurements" contaminated by noise. This noise has an amplitude that is uniformly distributed on the interval that is $\pm 0{\cdot}1\%$ of the 'true' measurement value, and is of random phase. Wave decomposition is performed using these contaminated measurements, and the wave amplitude estimates compared with their true values. This approach is taken in all subsequent simulations, although in some cases the relative amplitude of the noise is changed to aid visualization. The calculated wave amplitudes, normalized with respect to the true amplitudes, together with the condition number of the transformation matrix, **F**, are shown in Figure 5. It can be seen that the estimate of the wall-dominated wave is not affected significantly by this level of noise, except in the poorly conditioned region that occurs at low frequencies. In contrast, the estimate of the fluid-dominated wave is severely affected by this noise, to the extent that it is obviously not practical to estimate these wave amplitudes with this form of sensor array. However, the reduced sensitivity to measurement error with improving conditioning is readily apparent.

5.2. CASE 2: $k_1 \approx k_2$

Using the sensor placement given in equation (18), if $k_1 \approx k_2$ then $(z_2 - z_1) \approx (z_4 - z_3)$ and $|(z_3 - z_1)| \ll |(z_2 - z_1)|$. Term S in equation (17) therefore has a very slow variation with wavenumbers k_1 and k_2, while the more rapid variations (as opposed to modulation) of the remaining terms cancel at low frequencies, as seen in Figure 6. The variation of determinant with wavenumber at low frequencies is thus dominated by the slow modulation of the individual terms. The first zero in the determinant approximately coincides with the first zero in the third term, S (i.e., when $(k_1(z_1 - z_3)) \approx$

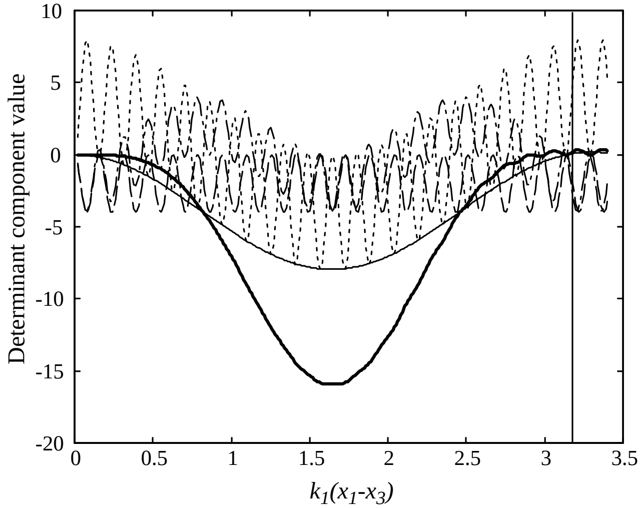

Figure 6. Terms contributing to determinant as a function of sensor spacing ($k_1 \approx k_2$): (——) Det(**F**); (– – – –) Q; (–·–·–) R; (——) S; (. . . . .) T.

$(k_2(z_1 - z_3)) \approx \pi)$. Since $|(z_3 - z_1)| \ll |(z_2 - z_1)|$ the first singularity of the transformation matrix occurs at a much higher wavenumber (and hence higher frequency) than would be the case if each wave type were to be sensed separately with the corresponding pair of sensors (i.e., when $k_1(z_2 - z_1) \approx k_2(z_4 - z_3) \gg \pi$). However, this frequency is very sensitive to the relative placement of the sensors, as small changes in $(z_2 - z_1)$ or $(z_4 - z_3)$ result in a relatively large change in $(z_3 - z_1)$.

An example of this situation is an uPVC pipe, having a 50 mm internal diameter and 2.5 mm wall thickness, containing helium. Under these circumstances $k_f \approx 1{\cdot}5k_e$. Letting the separation of the inner sensor pair $(z_2 - z_1) = 1{\cdot}333$ m and the separation of the outer sensor pair $(z_4 - z_3) = 2{\cdot}0$ m satisfies the initial assumption that $k_f(z_2 - z_1) \approx k_e(z_4 - z_3)$. In Figure 7 is shown the variation of the condition number of the transformation matrix with frequency, together with a plot of $1 + 1/|\det(\mathbf{F})|$, for this sensor array. As in the previous case, it can be seen that the determinant of the transformation matrix, **F**, can be used as a qualitative indicator of matrix condition.

As predicted from examination of the expression for the determinant of matrix, **F**, the proposed sensor array has a relatively wide frequency range, with the first singularity occurring at a frequency of approximately 1500 Hz, corresponding to $k_f|(z_3 - z_1)| \approx 3{\cdot}1$. The effect of increasing the distance between adjacent outer and inner sensors on the conditioning can be seen in Figure 8. Both decreasing the separation of the inner sensor pair and increasing the separation of the outer pair reduce the upper frequency limit of the array. However, in both of these cases the improvement in conditioning at low frequencies is relatively small.

The calculated wave amplitudes, normalized with respect to the true amplitudes, when random noise is added to the measurement as described in the previous subsection, are shown in Figure 9, together with the condition number of the transformation matrix. Again it can be seen that the estimate of the wall-dominated wave is not greatly affected by this level of noise, except in the poorly conditioned region that occurs at low frequencies. The estimate of the fluid-dominated wave is more severely affected, however, with the correlation between the condition number and sensitivity to measurement errors readily apparent.

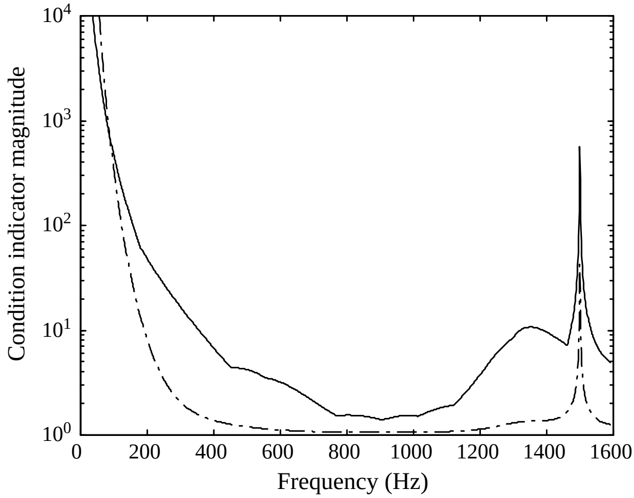

Figure 7. Estimates of matrix condition (uPVC/helium): (——) two-norm condition number; (–·–·–·–) $1 + 1/|\det(\mathbf{F})|$.

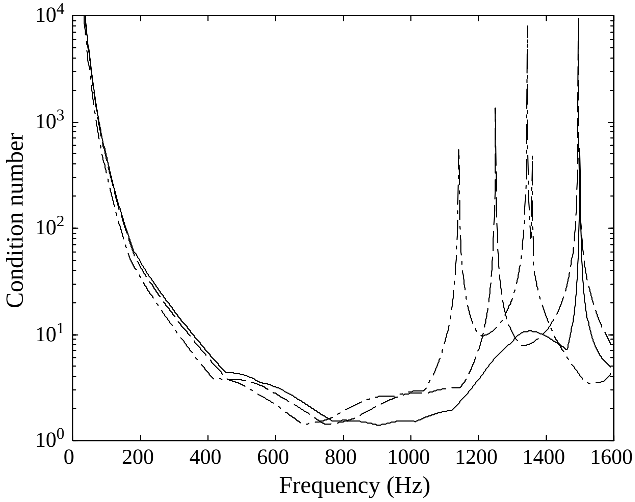

Figure 8. Matrix condition number (two-norm) (uPVC/helium $k_f \approx 1{\cdot}5k_e$): (——) $(z_2 - z_1) = 1{\cdot}333$ m, $(z_4 - z_3) = 2{\cdot}000$ m; (- - - - -) $(z_2 - z_1) = 1{\cdot}200$ m, $(z_4 - z_3) = 2{\cdot}000$ m; (–·–·–) $(z_2 - z_1) = 1{\cdot}333$ m, $(z_4 - z_3) = 2{\cdot}200$ m.

5.3. CASE 3: $k_2 = k_1/3$

In the special case when $k_2 = k_1/3$, if the sensor spacing proposed in equation (18) is used, a uniformly spaced sensor array is obtained and

$$\begin{aligned}\det(\mathbf{F}) = & -4[\sin(k_1(z_2 - z_1))\sin(3k_2(z_2 - z_1)) \\ & + \sin(3k_1(z_2 - z_1))\sin(k_2(z_2 - z_1)) + 2\sin(-k_1(z_2 - z_1))\sin(-k_2(z_2 - z_1)) \\ & - 2\sin(2k_1(z_2 - z_1))\sin(2k_2(z_2 - z_1))]. \end{aligned} \tag{19}$$

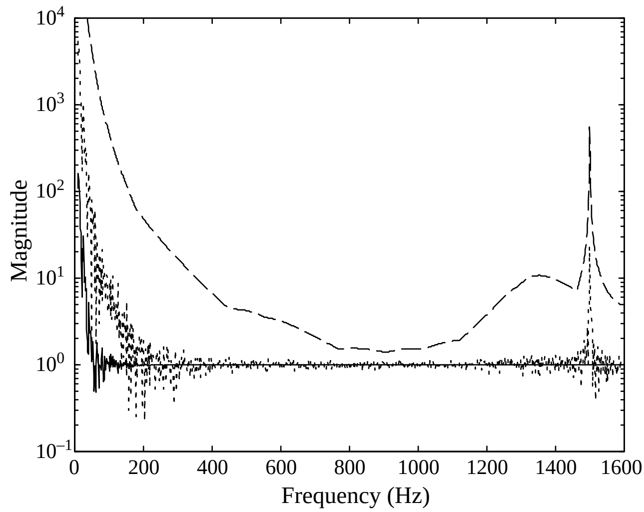

Figure 9. Condition number and normalized wave amplitude estimate (uPVC/helium): (——) extensional wave; (······) fluid wave; (--------) condition number.

The first zero in the determinant then occurs when $k_1(z_2 - z_1) = k_2(z_4 - z_3) = \pi$, as would be found if each wave type were sensed independently with a single pair of sensors.

5.4. THE EFFECTS OF WEAK COUPLING

A fundamental problem in the use of a single parameter sensor array is the fact that at least one vibration type is often relatively weakly coupled to the measured parameter. In this situation, the contribution of that vibration type to the measurements will often be relatively small. The estimation of the amplitudes of these waves is then sensitive to measurement errors, even if the transformation matrix, **F**, is well conditioned, because the levels of error in the measurements are significant relative to the contribution of the wave. This is apparent in the previous examples, where the estimate of the fluid-dominated wave is strongly affected by relatively low levels of noise. Under these circumstances, the simultaneous estimation of the two wave types becomes of questionable value. If the measured parameter is dominated by one vibration type, then the assumption of a single wave type being present is a valid approximation. A reasonable estimate of the amplitudes of these waves could thus be obtained using a single pair of sensors, with the benefits of improved conditioning, and reduced equipment and computational requirements. If the noise levels are comparable to the contribution of the 'minor' wave type, it will not be possible to get a reliable estimate of these wave amplitudes with the proposed sensors, irrespective of whether a two-sensor or a four-sensor array is used. In summary, an array of four identical sensors is appropriate for wave decomposition in this application only if both vibration types make a significant contribution to the measured variable. In practice, however, the use of different sensor types in the array (hybrid sensor arrays) offers the potential of improved wave amplitude estimates.

6. WAVE AMPLITUDE ESTIMATION USING A HYBRID SENSOR ARRAY

This section considers a sensor array that uses two different sensor types, meaning that, under the assumption of utilizing pairs of sensors, ψ_1 and ψ_2 in equation (12) are one variable measured at different locations, while ψ_3 and ψ_4 represent a second variable, again measured at different locations. Under these circumstances, the constants $q_{ij}^{\pm}$ will, in general, be different for each wave type and each measured variable. The value of the determinant, and therefore the conditioning of the wave amplitude estimate, then depends not only on the placing of the sensors but also on the relative values of $q_{ij}^{\pm}$. Good conditioning, and a relatively broad operating frequency range, will be achieved if the determinant is dominated by a term having a large magnitude and relatively slow variation with wavenumber. This requirement is fulfilled if $|q_{11}^{\pm}||q_{22}^{\pm}| \gg |q_{12}^{\pm}||q_{21}^{\pm}|$. If the sensors are placed such that $k_1(z_2 - z_1) \approx k_2(z_4 - z_3)$, the determinant is then dominated by the first term of equation (14) or (15), and the first zero of the determinant occurs when $k_1(z_2 - z_1) \approx k_2(z_4 - z_3) = \pi$, irrespective of the relative values of k_1 and k_2. This provides a better-conditioned problem than is achieved using identical sensors, together with an acceptable operating frequency range.

In physical terms, the requirement that $|q_{11}^{\pm}||q_{22}^{\pm}| \gg |q_{11}^{\pm}||q_{22}^{\pm}|$ means selecting, for each vibration type of interest, a measured variable that is strongly influenced by that vibration type while being weakly influenced by the other. This is more apparent when the relationship is expressed as $|q_{11}^{\pm}| \gg |q_{12}^{\pm}|$ and $|q_{22}^{\pm}| \gg |q_{21}^{\pm}|$, which can be obtained through the use of appropriate units for the variables, for example expressing pressure in terms of MPa rather than Pa. (It should be noted that changing the units of a measured variable and wave amplitude has the effect of multiplying the affected columns of the matrix **F** by the relevant scaling factor, and the affected rows by the inverse of that scaling factor, and thus does not change the value of the determinant.) In the limiting case, the transformation matrix, **F**, becomes block diagonal and the amplitudes of the different wave types can be estimated independently.

The effect of reducing the coupling between the measured variables on the conditioning can be seen in Figure 10. This replicates the simulation of the single parameter array for $k_1 \gg k_2$ (section 5.1), but in this case it is assumed that a hybrid array is used. The measured variables and wave amplitude descriptors are chosen such that $|q_{11}^{\pm}| = |q_{22}^{\pm}| = 1$, $|q_{12}^{\pm}| = |q_{21}^{\pm}| = c$, where $c \leqslant 1$ and determines the degree of coupling between the measured variables. Figure 10 shows the condition number of the matrix **F** for a variety of values of c, starting with $c = 1$ (i.e., a single-parameter array). It is apparent that a hybrid array that offers even a moderate reduction in coupling over a single-parameter array can give substantially improved conditioning and increased frequency range.

Clearly, if measured variables can be chosen such that c is very small then it is possible to measure parameters which are not strongly coupled, and under these circumstances the simultaneous estimation of the two wave types *may* not be required. However, even in the presence of weak coupling it is possible for both wave types to contribute significantly to the measured variables due to the relative amplitudes of the waves. The simultaneous estimation of wave amplitudes is therefore recommended unless a priori knowledge of the system indicates otherwise.

The simulated results of wave decomposition using a hybrid array and a single parameter array are shown in Figure 11(a) and (b). The simulation involves an air-filled steel pipe (as used in section 5.1), and the hybrid array measures axial velocity and air pressure, while axial motion is measured in the single-parameter array. As in the previous examples, random noise is added to the simulated measurements (in this case uniformly distributed on the interval $\pm 0{\cdot}5\%$) and the resulting wave amplitude estimates normalized with respect to the true values. The upper row of Figure 11 shows the amplitude estimates for the fluid-dominated wave, while the lower row shows the estimates for the wall-dominated wave. It can be seen that both arrays give reasonable results for the wall-dominated wave, with the hybrid array giving much better performance at lower

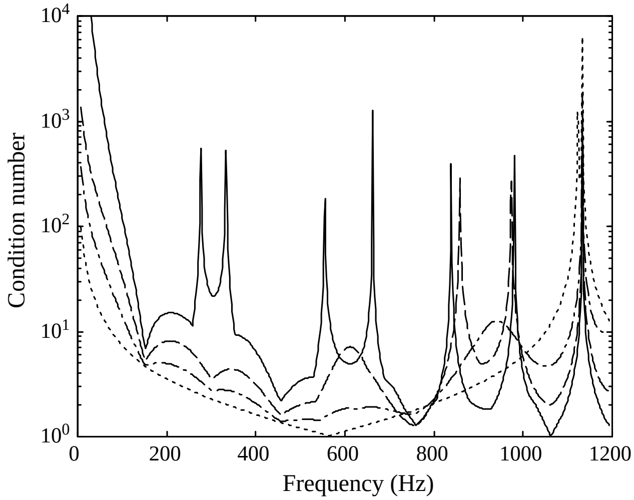

Figure 10. Matrix condition number (steel/air)(hybrid array): (——) $c = 1$; (– – – –) $c = 0{\cdot}5$; (–·–·–) $c = 0{\cdot}2$; $c = 0{\cdot}01$.

frequencies and a much greater frequency range. It is apparent, however, that only the hybrid array gives a reasonable estimate for the fluid-dominated wave.

In many practical situations, direct pressure measurements, while desirable, are not possible. Under these circumstances an alternative hybrid array is required. One possibility substitutes the pressure measurements with measurements of circumferential strain, as proposed by Pinnington and Briscoe [4]. For comparative purposes the results achieved with such an array, under the conditions described above, are shown in Figure 11(c). It is clear that this array gives better results than the single-parameter array, but is significantly more sensitive to measurement noise than the pressure/axial velocity array when estimating the amplitude of the fluid-dominated wave. This can be explained by the fact that, with the frequencies and amplitudes used in the simulation, the circumferential strain is dominated by the (predominantly) extensional wave. As a consequence, the levels of error in the measurements are significant relative to the contribution of the fluid-dominated wave, resulting in the situation described in section 5.4. In this specific case the strain sensors, being placed to sense the fluid wave effectively, are closely spaced relative to the wavelength of the dominant wave. They, therefore, measure almost identical quantities, and in particular have very similar phase. Calculations of relative phase and difference in magnitude of these measurements, upon which the estimation of fluid wave amplitude is heavily reliant, are thus prone to large relative errors.

7. CONCLUDING REMARKS

The sensitivity of the wave amplitude estimate to measurement and other errors is determined by the conditioning of the transformation matrix. In designing a sensor array with which to estimate wave amplitudes, it is therefore desirable to obtain a well-conditioned transformation matrix. However, the relationship between sensor type and placement and matrix condition is not obvious when more than one vibration type affects

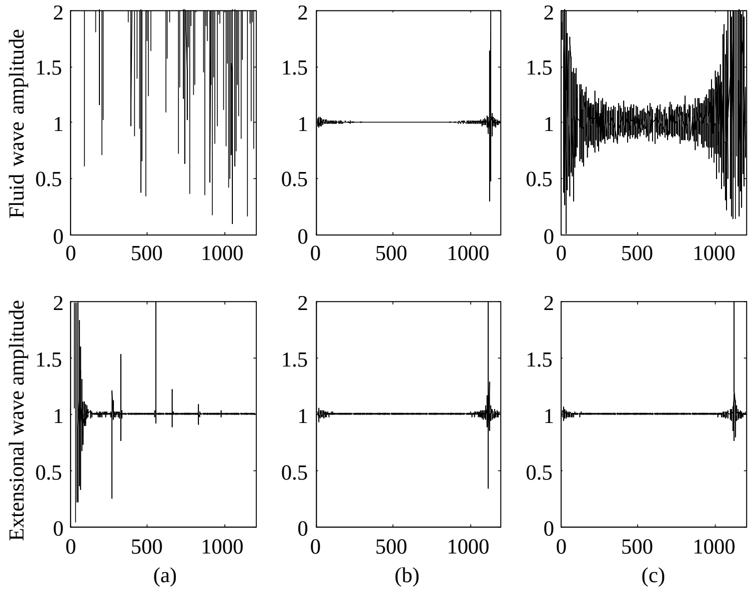

Figure 11. Normalized wave amplitude estimate (steel/air): (a) single-parameter array; (b) hybrid pressure/axial velocity array; (c) hybrid circumferential strain/axial velocity array.

the measured variables. Under these circumstances, study of the analytical expression for the determinant of the transformation matrix can give improved insight into sensor array design. In this paper, the simultaneous estimation of two wave types is discussed. Particular reference is made to coupled axial motion and pressure pulses in a fluid-filled pipe, but the approach is completely general.

If two vibration types significantly affect the possible measured variables, it is preferable that two pairs of sensors are used, each pair measuring a different variable. The first sensor pair is chosen to sense the first vibration type effectively, with a minimum of influence from the second vibration type. The measured variable for the first sensor pair should therefore be strongly influenced by the first vibration type but weakly influenced by the second, and the separation of this pair should be appropriate for the wavelength of the first wave type. The second sensor pair and its placement are chosen on a similar basis, but with the intention of sensing the second vibration type effectively. In the limiting case, the amplitudes of the waves associated with the different vibration types can be estimated independently.

If an array of identical sensors is used and the measured variables are used to describe the wave amplitudes, the operational frequency range of the sensor array is strongly dependent on the relative size of the wavenumbers of the two wave types. If the two wavenumbers are similar in magnitude, then the first singularity of the transformation matrix occurs at a relatively high frequency if the sensor placement given by equation (18) is used. However, this frequency is very sensitive to changes in the placement of the sensors.

If the wavenumbers differ greatly, then the frequency at which the first singularity occurs is relatively low using the sensor placement given by equation (18), restricting the frequency range of the sensor array. The frequency of singularity can be increased through decreasing the overall span of the sensor array, at the expense of poorer conditioning at low frequencies.

Irrespective of the conditioning achieved, it is necessary to ensure that the noise levels in the measurements are small, not just relative to the measurements themselves, but small relative to the contributions of the individual waves to the measurements. This applies to both single-parameter and hybrid arrays.

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APPENDIX A: EQUATIONS OF MOTION FOR A FLUID-FILLED PIPE

The vibration of fluid-filled pipes has been studied by a number of authors (e.g., references [5, 6]). It is, however, necessary to make a number of simplifying assumptions in order to yield approximate equations of motion that are amenable to analysis, and whose solutions may be related to the measurements of a practical (i.e., relatively simple) measurement system. Thus, only certain models are appropriate for this task. The theory developed in this section follows the method proposed by de Jong [7]. It is assumed that the frequencies of interest are well below the ring frequency of the pipe, being below the cut-on frequency of the $n = 2$ circumferential mode. At these frequencies the motion of the pipe is dominated by the $n = 0$ and 1 modes, and there is little deformation of the pipe cross-section. Membrane stresses are, therefore, the dominant stresses in the pipe, and local bending and radial shear deformation can be neglected. Under these circumstances, it can be shown [7] that the equations of motion can be approximated by

$$\frac{\mathrm{d}\bar{M}_z}{\mathrm{d}z} = -\rho_s J_s \omega^2 \bar{\phi}_z, \quad \frac{\mathrm{d}\bar{\phi}_z}{\mathrm{d}z} = \frac{\bar{M}_z}{G_s J_s}, \tag{A1a, b}$$

$$\frac{\mathrm{d}\bar{p}}{\mathrm{d}z} = \rho_f \omega^2 \bar{v}_f, \quad \frac{\mathrm{d}\bar{v}_f}{\mathrm{d}z} = -\frac{1}{K_f}\left(1 + \frac{2rK_f}{hE_s}\right)\bar{p} + 2\nu \frac{r}{r_i}\frac{\bar{F}_z}{E_s A_s},$$

$$\frac{\mathrm{d}\bar{F}_z}{\mathrm{d}z} = -\rho_s A_s \omega^2 \bar{u}_z, \quad \frac{\mathrm{d}\bar{u}_z}{\mathrm{d}z} = \frac{\bar{F}_z}{E_s A_s} - 2\nu \frac{r}{r_i}\frac{A_f \bar{p}}{E_s A_s}. \tag{A2a-d}$$

$$\frac{\mathrm{d}\bar{F}_x}{\mathrm{d}z} = -m\omega^2 \bar{u}_x, \quad \frac{\mathrm{d}\bar{u}_x}{\mathrm{d}z} = \frac{\bar{F}_x}{\kappa_s G_s A_s} - \bar{\phi}_y, \quad \frac{\mathrm{d}\bar{M}_y}{\mathrm{d}z} = -\bar{F}_x - \rho_s I_s \omega^2 \bar{\phi}_y, \quad \frac{\mathrm{d}\bar{\phi}_y}{\mathrm{d}z} = \frac{\bar{M}_y}{E_s I_s}. \tag{A3a-d}$$

$$\frac{\mathrm{d}\bar{F}_y}{\mathrm{d}z} = -m\omega^2 \bar{u}_y, \quad \frac{\mathrm{d}\bar{u}_y}{\mathrm{d}z} = \frac{\bar{F}_y}{\kappa_s G_s A_s} - \bar{\phi}_x, \quad \frac{\mathrm{d}\bar{M}_x}{\mathrm{d}z} = \bar{F}_y - \rho_s I_s \omega^2 \bar{\phi}_x, \quad \frac{\mathrm{d}\bar{\phi}_x}{\mathrm{d}z} = \frac{\bar{M}_x}{E_s I_s}. \tag{A4a-d}$$

The meanings of the symbols are given in Appendix B and the sign conventions for displacements ($\bar{u}$, $\bar{v}_f$), rotations ($\bar{\phi}$), forces ($\bar{F}$), moments ($\bar{M}$), and pressure ($\bar{p}$) are as defined in Figure A1. Note that the overbar indicates that these quantities are averaged over the pipe cross-section.

The equations of motion can be considered as describing four independent (i.e., uncoupled) motions, and therefore four independent vibration types. Equations (A1a,b) and (A2a–d) describe the torsional motion of the pipe and the axial motion of the pipe and fluid respectively, while equations (A3a–d) and (A4a–d) describe the transverse motion of the pipe and fluid in the $x-z$ and $y-z$ planes respectively. As these motions are independent they may be considered, and measured, separately. This paper considers only equation (A2a–d), describing axial motion of the pipe and fluid, as an example of coupled motion.

Assuming that the possible solutions to the equations of motion can be written in terms of propagating and evanescent waves, each possible solution will have the form

$$\psi(z,t) = \Phi e^{\lambda z} e^{i\omega t}, \tag{A5}$$

and the general solution will be given by

$$\psi(z,t) = \sum_{n=1}^{N} \Phi_n e^{\lambda_n z} e^{i\omega t}, \tag{A6}$$

i.e., the sum of a number of different waves.

Assuming that the dynamic quantities vary spatially with $e^{\lambda z}$, combining the four equations (A2a–d) gives the dispersion equation

$$\lambda^4 - \lambda^2(\hat{k}_f^2 + \hat{k}_e^2) + \hat{k}_f^2 \hat{k}_e^2 \left(1 - \left(2v\frac{r}{r_i}\right)^2 \sigma\right) = 0, \tag{A7}$$

where λ are the wavenumber solutions to the equation,

$$\hat{k}_e^2 = \frac{\rho_s \omega^2}{E_s}, \quad \hat{k}_f^2 = \frac{\rho_f \omega^2}{K_f}(1 + 2rK_f/hE_s), \tag{A8a, b}$$

and

$$\sigma = \frac{A_f K_f}{A_s E_s} \frac{1}{(1 + 2rK_f/hE_s)}. \tag{A8c}$$

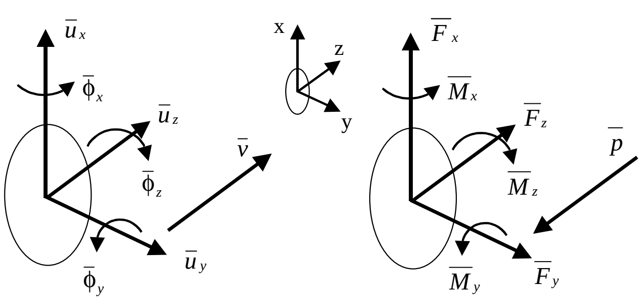

Figure A1. Conventions for displacement, rotation, forces, moments, and pressure.

Since the dispersion equation is of fourth order in λ, there are four possible wavenumber solutions. For the pipe geometry being considered ($h \ll r$, damping negligible) the resulting λ are two purely imaginary conjugate pairs, corresponding to two (positive and negative going) propagating wave pairs. For typical fluids and pipe materials, the wavenumber pair with the greatest magnitude ($\pm ik_f$) corresponds to a vibration dominated by a pressure pulse in the fluid, but also resulting in a circumferential strain, owing to the finite modulus of elasticity of the pipe wall, and axial strain due to the Poisson effect. The wavenumber of this wave pair is therefore similar to that for a pressure pulse in a rigid pipe, but somewhat higher owing to the elasticity of the pipe reducing the wave speed. The wavenumber pair with the lower magnitude ($\pm ik_e$) corresponds to a vibration dominated by axial motion of the pipe wall, but also resulting in circumferential strain (due to the Poisson effect) and, consequentially, pressure pulses in the fluid. The wavenumber for this wave pair is therefore similar to that for axial vibration in a rod, but somewhat lower owing to the resistance to the Poisson effect endowed by the fluid.

As discussed in section 2, the amplitude of a wave may be expressed in terms of any variable that is affected by the presence of that wave. For a particular wave, there exist fixed relationships between the different response variables. These relationships can be determined for axial motion of the pipe and fluid by combining the relation in equation (A2a–d). For example, the various response parameters can be expressed in terms of the pressure as

$$\bar{u}_z = \left(\frac{\lambda a_i}{2va\rho_f \omega^2} + \frac{a_i}{2vaK_f} \left(1 + \frac{2aK_f}{hE_s} \right) \frac{1}{\lambda} - 2v \frac{a}{a_i} \frac{A_f}{E_s A_s} \frac{1}{\lambda} \right) \bar{p},$$

$$\bar{v}_f = \frac{\lambda}{\rho_f \omega^2} \bar{p}, \quad \bar{F}_z = \frac{E_s A_s a_i}{2va} \left(\frac{\lambda^2}{\rho_f \omega^2} + \frac{1}{K_f} \left(1 + \frac{2aK_f}{hE_s} \right) \right) \bar{p},, \qquad \text{(A9a–c)}$$

where λ is the wavenumber solution appropriate for the particular wave. A measured variable related to axial motion of the pipe and fluid can therefore be expressed with complete generality as

$$\psi(z) = q_f^+ \Phi_f^+ e^{-ik_f z} + q_f^- \Phi_f^- e^{ik_f z} + q_e^+ \Phi_e^+ e^{-ik_e z} + q_e^- \Phi_e^- e^{ik_e z}, \qquad \text{(A10)}$$

where q_f^+ is the conversion factor that relates the parameter used to describe the wave amplitude to the measured variable for the positive-going fluid-dominated wave, and similarly for the other waves.

APPENDIX B: NOMENCLATURE

A_f	fluid area
A_s	pipe wall cross-sectional area
E_s	elastic modulus of pipe wall
$\bar{F}$	force
$\mathbf{F}$	transformation matrix relating measured variables to wave amplitudes
G_s	shear modulus of pipe wall
h	pipe wall thickness
J_s	torsional second moment of area of pipe
k_e	wavenumber of (predominantly) extensional wave
$\hat{k}_e$	approximation of k_e
K_f	wavenumber of (predominantly) pressure wave
$\hat{k}_f$	approximation of k_f

K_f	bulk modulus of fluid
m	linear density of pipe + fluid
$\bar{M}$	moment
$\bar{p}$	pressure
Q	conversion factor between units of measured variable and units of wave amplitude
r	pipe radius to mid-thickness
r_I	internal pipe radius
$\bar{u}$, $\bar{v}_f$	displacements
$\bar{\phi}$	rotation
$\kappa(\mathbf{F})$	2-norm condition number of matrix **F**
λ	wavenumber solutions to dispersion equation
ν	the Poisson ratio for pipe wall
ρ_f	fluid density
ρ_s	solid (pipe wall) density
σ	pipe/fluid system parameter
ω	angular frequency
ψ	measured variable
$\boldsymbol{\psi}$	vector of measured variables
Φ	wave amplitude
$\boldsymbol{\Phi}$	vector of wave amplitudes
	An overbar indicates that a quantity is averaged over the pipe cross-section.

Subscripts

e	pertaining to extensional wave
f	pertaining to fluid or fluid wave
s	pertaining to solid (pipe wall)